REWRITING SYSTEMS IN ALTERNATING KNOT GROUPS WITH THE DEHN PRESENTATION

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ABSTRACT. Every tame, prime and alternating knot is equivalent to a tame, prime and alternating knot in regular position, with a common projection. In this work, we show that the Dehn presentation of the knot group of a tame, prime, alternating knot, with a regular and common projection has a finite and complete rewriting system. Although there are rules in the rewriting system with left-hand side a generator and which increase the length of the words we show that the system is terminating.

1. INTRODUCTION

In [2], we showed that the augmented Dehn presentation of the knot group of a tame, prime, alternating knot in regular position, with a common (elementary) projection [7, p.267] has a finite and complete rewriting system (with no need of completion) and this result holds also for alternating links satisfying our assumptions. We showed there that there are exactly two such rewriting systems and we gave an algorithm which finds them. This was carried out using graph theory, applied to the projection of the knot and to the projection's dual graph. In this paper, we use one of the complete rewriting systems for the augmented Dehn presentation of the knot group to find a finite and complete rewriting system for the Dehn presentation. The same process can be applied on the second rewriting system.

The main idea is that we can divide the generators in the Dehn presentation of the knot group and their inverses in two disjoint sets, called the sources and the sinks. This division of the generators is derived from the algorithm which finds the complete rewriting system for the augmented Dehn presentation of the knot group and it permits us to compare the words in the free group generated by these generators in an efficient way.

We will describe in some words the way we do this and we refer the reader to section 2 for more details. If K is a knot and $\wp(K)$ its projection on R^2, then $\wp(K)$ is said to be **common** if the boundaries of any two distinct domains have at most one edge in common and each crossing is on the boundary of exactly four distinct domains [3][9][7, p267]. In the **Dehn presentation** of the knot group (the fundamental group of $R^3 \setminus K$) the generators are the domains of $R^2 \setminus \wp(K)$, labelled by $x_0, x_1, ..., x_n$, starting with the unbounded domain (see Fig.1). The relations arise from the crossings: at each crossing, four distinct regions x_a, x_b, x_c, x_d meet and the relation arising from this crossing is r: $x_a \overline{x_b} x_c \overline{x_d} = 1$, where $\overline{\mathbf{x}}$ **denotes the inverse of** x.

The **Dehn presentation** of the fundamental group of $R^3 \setminus$ K is $<$ $x_0, x_1, ..., x_n \mid r_1, ..., r_{n-1}, x_0 >$ where each r_m has the form $x_i \overline{x_j} x_k \overline{x_l} = 1$ with x_i, x_j, x_k, x_l generators and the **augmented Dehn presentation** is the Dehn presentation with the relation $\{x_0 = 1\}$ deleted, i.e $< x_0, x_1, ..., x_n \mid r_1, ..., r_{n-1} >$.

Weinbaum in [9] showed that the augmented Dehn presentation given by K presents the free product of the knot group $\pi(R^3 \setminus K)$ and an infinite cyclic group.

The way to divide the generators in the two disjoint sets of sources

and sinks can be described easily as follows: choose x_0 to be a source, then all its neighbours will be sinks and so on iteratively. Next, if x_0 is a source, then $\overline{x_0}$ is also chosen to be a source and the same holds for all the generators. This process is consistent, since when the knot projection is regular a chess-boarding colouring of the domains is possible (a regular knot projection is a planar graph in which every vertex has even degree 4). The generators are then renamed according to their being a source or a sink: if x_* is a source then it is renamed s_* and if x_* is a sink then it is renamed t_* and the relations and the rules are "rewritten" accordingly. For the eight knot group, if x_0 and $\overline{x_0}$ are sources, then x_1, x_2, x_3, $\overline{x_1}$, $\overline{x_2}$, $\overline{x_3}$ are sinks and x_4, x_5, $\overline{x_4}$, $\overline{x_5}$ are also sources (see Fig.1). So,the main result of this paper can be stated as follows:

Theorem. *Let K be a tame, prime and alternating knot with a regular and common projection. Then the Dehn presentation of the knot group of K has a complete and finite rewriting system.*
We shall denote this rewriting system of K by $\Re''$.

Example: The augmented Dehn presentation of the eight knot group (see Fig.1)

$$< s_0, t_1, t_2, t_3, s_4, s_5 \mid t_1\overline{s_0}t_2\overline{s_4}, t_2\overline{s_0}t_3\overline{s_5}, t_1\overline{s_5}t_3\overline{s_0}, t_2\overline{s_5}t_1\overline{s_4} >$$

In Section 2, we give some definitions concerning rewriting systems. Then,we give a brief survey of the results obtained in [2] and refer the reader for more details. In Section 3, we find a rewriting system $\Re'$ for the Dehn presentation of the knot group and show some of its properties and then we find a rewriting system $\Re''$ which is equivalent to $\Re'$. In Section 4, we show that the rewriting system $\Re''$ is complete.

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The main difficulty is to show that it is terminating, since there are rules in $\Re''$ which increase the length of the words. **In this paper, we shall assume that knots are : tame, prime and alternating with a regular and common projection.**

2. Preliminaries

2.1. **On rewriting systems.** The terminology in this section is from [1] and [4]. Let $\sum$ be a non-empty set. We denote by $\sum^*$ the free monoid generated by $\sum$; elements of $\sum^*$ are finite sequences called **words** and the empty word will be denoted by 1. The length of the sequence is called the **length of the word** w and is denoted by $\ell(w)$.

Definition. *A **rewriting system** $\Re$ on $\sum$ is a set of ordered pairs in $\sum^* \times \sum^*$.*

*If $(l, r) \in \Re$ then for any words u and v in $\sum^*$, we say the word ulv **reduces** to the word urv and we write $ulv \rightarrow urv$.*

A word w is said to be **reducible** if there is a word z such that $w \rightarrow z$. If there is no such z we call w **irreducible**.

A rewriting system $\Re$ is called **terminating (or Noetherian)** if there is no infinite sequence of reductions $w_1 \rightarrow w_2 \rightarrow ... \rightarrow w_n \rightarrow ...$.

$\Re$ is called **locally confluent** if for any words u, v, w in $\sum^*$, $w \rightarrow u$ and $w \rightarrow v$ implies that there is a word z in $\sum^*$ such that $u \rightarrow^* z$ and

$v \to^* z$, where " $\to^*$ " denotes the reflexive transitive closure of the relation " $\to$ " .

We call the triple of non-empty words u, v, w in $\sum^*$ an **overlap ambiguity** if there are r_1, r_2 in $\sum^*$ such that $uv \to r_1$ and $vw \to r_2$ are rules in $\Re$. We then say that $r_1 w$ and $u r_2$ are the corresponding **critical pair.** When the triple u, v, w in $\sum^*$ is an overlap ambiguity we will say that the rules $uv \to r_1$ and $vw \to r_2$ **overlap at** v or that there is an **overlapping** between the rules $uv \to r_1$ and $vw \to r_2$. If there exists a word z such that $r_1 w \to^* z$ and $u r_2 \to^* z$, then we say that the critical pair resulting from the overlapping of the rules $uv \to r_1$ and $vw \to r_2$ in $\Re$ **resolves**.

The triple u, v, w of possibly empty words in $\sum^*$ is called an **inclusion ambiguity** if there are r_1, r_2 in $\sum^*$ (which must be distinct if both u and w are empty, but otherwise may be equal) such that $v \to r_1$ and $uvw \to r_2$ are rules in $\Re$. We then say that $u r_1 w$ and r_2 are the corresponding **critical pair**. If there exists a word z such that $u r_1 w \to^* z$ and $r_2 \to^* z$, then we say that the critical pair resulting from the inclusion ambiguity of the rule $v \to r_1$ in $uvw \to r_2$ in $\Re$ **resolves**.

$\Re$ is called **complete (or convergent)** if $\Re$ is terminating and locally confluent or in other words if $\Re$ is terminating and all the critical pairs resolve. So if $\Re$ is complete then every word w in $\sum^*$ has a unique irreducible equivalent word z, which is called the **normal form** of w.

We say that two rewriting systems $\Re$ and $\Re'$ are **equivalent** if: $w_1 \leftrightarrow^* w_2$ modulo $\Re$ if and only if $w_1 \leftrightarrow^* w_2$ modulo $\Re'$, where $\leftrightarrow^*$ denotes the equivalence relation generated by $\to$.

We call a rewriting system $\Re$ **reduced** if for any rule $l \to r$ in $\Re$, r is irreducible and there is no rule $l' \to r'$ in $\Re$ such that l' is a subword of l. If $\Re$ is complete then there exists a reduced and complete rewriting system $\Re'$ which is equivalent to $\Re$ [8]. The reduced rewriting system $\Re'$ which is equivalent to $\Re$ is obtained in two steps: (1) whenever $l \to r$ is a rule in $\Re$ and r is not irreducible we reduce r to its normal form. (2) whenever $l \to r$, $l' \to r'$ are rules in $\Re$ and l' is a subword of l we remove from $\Re$ the rule $l \to r$.

2.2. **Preliminary results.** Let G be a group presented by $< X \mid R >$, where X is a set of free generators $\{x_0, x_1, ..., x_n\}$ and R the set of relations. In order to define a rewriting system for G, we have to consider the **monoid presentation** of G, $< X \cup \overline{X} \mid R \cup R_0 >$, where $\overline{X}$ denote the set $\{\overline{x_0}, ..., \overline{x_n}\}$ and $R_0 = \{x_i \overline{x_i} = 1, \overline{x_i} x_i = 1$, with $i \in \{0, 1, ..., n\}\}$. The **symmetrization process on a relator** r is the following process: we consider all the relations which can be obtained as consequences of $r, \overline{r}$ and all their cyclic permutations in a group and we define $S(r)$ to be the minimal set of relations from which all the other relations can be derived in the monoid. Example: if $r : ab = 1$ where $a, b \in X \cup \overline{X}$ then we obtain from the symmetrization process the set of relations $\{ab = 1, ba = 1, \overline{b}\,\overline{a} = 1, \overline{a}\,\overline{b} = 1, b = \overline{a}, a = \overline{b}\}$ and $S(r) = \{a = \overline{b}, b = \overline{a}\}$.

We refer the reader to [2] where this process is done in detail for the relations in the knot group and also to [6]. In what follows, we will state results from [2] without proofs.

Remark 2.1. *For each relator* $r_m : x_i \overline{x_j} x_k \overline{x_l} = 1$, *the set* $S(r_m)$ *is the following set of relations:*

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$$x_i\overline{x_j} = x_l\overline{x_k}$$

$$\overline{x_j}x_k = \overline{x_i}x_l$$

$$x_k\overline{x_l} = x_j\overline{x_i}$$

$$\overline{x_l}x_i = \overline{x_k}x_j$$

The set of relations R', which is the union of the sets $\{S(r_m)\}$ for $1 \leq m$ $\leq n-1$ is equivalent to the set of relations $R = \{r_m \mid 1 \leq m \leq n-1\}$ for a knot group G, satisfying our assumptions and the problem is then how to orientate the relations in R' in order to define a rewriting system.

Example: the set of relations R' for the figure-eight knot group

$$
\begin{array}{llll}
s_0\overline{t_1} = t_2\overline{s_4} & s_0\overline{t_2} = t_3\overline{s_5} & s_0\overline{t_3} = t_1\overline{s_5} & s_4\overline{t_1} = t_2\overline{s_5} \\
\overline{s_0}t_2 = \overline{t_1}s_4 & \overline{s_0}t_3 = \overline{t_2}s_5 & \overline{s_0}t_1 = \overline{t_3}s_5 & \overline{s_4}t_2 = \overline{t_1}s_5 \\
s_4\overline{t_2} = t_1\overline{s_0} & s_5\overline{t_3} = t_2\overline{s_0} & s_5\overline{t_1} = t_3\overline{s_0} & s_5\overline{t_2} = t_1\overline{s_4} \\
\overline{s_4}t_1 = \overline{t_2}s_0 & \overline{s_5}t_2 = \overline{t_3}s_0 & \overline{s_5}t_3 = \overline{t_1}s_0 & \overline{s_5}t_1 = \overline{t_2}s_4
\end{array}
$$

Using the fact that all sides of relations in R' have length 2, we defined a graph Δ, called the derived graph, in the following way:

- The vertex-set is the set $X \cup \overline{X}$.
- For a, b in $X \cup \overline{X}$, there is an oriented edge $a \to b$ n Δ, if there is a relation in R' such that the word ab is one of its sides.

Example: The derived graph Δ for the figure-eight knot (see Fig.2)

A subgraph A of Δ is said to be an **antipath** in Δ, if each vertex of A is a source or a sink in A.The number of edges in this set is called the **length** of the antipath. Multiple edges (oriented the same) are not allowed in an antipath. An antipath need not be connected and an

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equivalent definition of an antipath is that no oriented path of length 2 occurs. An antipath A in Δ of length half the number of edges in Δ will define a rewriting system with no ambiguity overlap between the rules in the following way: all the edges in A will be the left-hand sides of the rules and all the edges in the complement of A will be the right-hand sides of rules.

Example: An antipath in the graph Δ for the figure-eight knot (see Fig.3)

It holds that Δ, the derived graph, is the disjoint union of two Eulerian closed paths of even length in which all the closed paths have even length. And as such, Δ admits two disjoint antipaths, each of length half the number of edges in Δ, which define a finite and complete rewriting system (with no need of completion). We will describe the algorithm to find one of the two antipaths $\hat{A}$, the one we will work with in this paper: choose x_0 and $\overline{x_0}$ to be sources in $\hat{A}$, which means that all the edges going out of x_0 and $\overline{x_0}$ will belong to $\hat{A}$ and this will determine $\hat{A}$ completely (in an iterative way). In fact, by having a glance at the knot projection , one can see immediately for each pair $(x_i, \overline{x_i})$ if these are sources or sinks in $\hat{A}$: in the example of the figure-eight knot if x_0 and $\overline{x_0}$ are sources in $\hat{A}$ then x_1, $\overline{x_1}$,x_2, $\overline{x_2}$, x_3 and $\overline{x_3}$ are sinks in $\hat{A}$ and $x_4, \overline{x_4}$, $x_5, \overline{x_5}$ are also sources in $\hat{A}$ (by a chess-boarding effect). We rename the generators according to their being a sink or a source in the antipath $\hat{A}$: if the generator x_* is a source it is denoted by s_* and if the generator $x_.$ is a sink it is denoted by $t_.$. We will denote by $\Re$ the rewriting system defined by the antipath $\hat{A}$, with the rules "rewritten" accordingly.

Example: The rewriting system $\Re$ for the figure-eight knot group:(see Fig.3)

$$s_0\overline{t_1} \to t_2\overline{s_4} \qquad s_0\overline{t_2} \to t_3\overline{s_5} \qquad s_0\overline{t_3} \to t_1\overline{s_5} \qquad s_4\overline{t_1} \to t_2\overline{s_5}$$

$$\overline{s_0}t_2 \to \overline{t_1}s_4 \qquad \overline{s_0}t_3 \to \overline{t_2}s_5 \qquad \frac{\overline{s_0}t_1 \to}{inet_3s_5} \qquad \overline{s_4}t_2 \to \overline{t_1}s_5$$

$$s_4\overline{t_2} \to t_1\overline{s_0} \qquad s_5\overline{t_3} \to t_2\overline{s_0} \qquad s_5\overline{t_1} \to t_3\overline{s_0} \qquad s_5\overline{t_2} \to t_1\overline{s_4}$$

$$\overline{s_4}t_1 \to \overline{t_2}s_0 \qquad \overline{s_5}t_2 \to \overline{t_3}s_0 \qquad \overline{s_5}t_3 \to \overline{t_1}s_0 \qquad \overline{s_5}t_1 \to \overline{t_2}s_4$$

$$s_i\overline{s_i} \to 1 \qquad i = 0,4,5 \qquad \overline{s_i}s_i \to 1 \qquad i = 0,4,5$$

$$t_i\overline{t_i} \to 1 \qquad i = 1,2,3 \qquad \overline{t_i}t_i \to 1 \qquad i = 1,2,3$$

Remark 2.2. *Each set $S(r_m)$, with $r_m : x_i\overline{x_j}x_k\overline{x_l}$, gives the following set of rules in $\Re$ if $x_i, x_k, \overline{x_i}, \overline{x_k}$ are sources and $x_j, x_l, \overline{x_j}, \overline{x_l}$ are sinks in the antipath $\hat{A}$ which defines $\Re$:*

$$(1) \quad s_i\overline{t_j} \to t_l\overline{s_k}$$
$$(2) \quad \overline{s_i}t_l \to \overline{t_j}s_k$$
$$(3) \quad s_k\overline{t_l} \to t_j\overline{s_i}$$
$$(4) \quad \overline{s_k}t_j \to \overline{t_l}s_i$$

If $x_j, x_l, \overline{x_j}, \overline{x_l}$ are sources and $x_i, x_k, \overline{x_i}, \overline{x_k}$ are sinks in the antipath $\hat{A}$, then a set of rules of the same kind is obtained. Note that since x_0 and $\overline{x_0}$ are sources in $\hat{A}$, x_0 and $\overline{x_0}$ will appear as the first letter of the left-hand side of a rule or as the last letter of the right-hand side.

3. DEFINITION OF A REWRITING SYSTEM $\Re'$ FOR THE DEHN PRESENTATION

3.1. **Definition of $\Re'$.**

We will use in what follows some tools developed in [9]. Let K be a tame, prime and alternating knot whose projection is regular and common. Let us denote the knot group $\pi(R^3 \setminus K)$ by G and assume its Dehn presentation is $< x_0, x_1, ..., x_n \mid$

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$r_1, ..., r_{n-1}, x_0 >$. So H , the group presented by the augmented Dehn presentation of $\pi(R^3 \setminus K)$ is $< x_0, x_1, ..., x_n \mid r_1, ..., r_{n-1} >$. Let $\Re$ be the complete and finite rewriting system for H obtained in the previous section. Let F be the free group on n+1 generators $x_0, x_1, ..., x_n$. Let Φ be the endomorphism of F determined by: $\Phi(x_0) = 1$ and $\Phi(x_j) = x_j, 1 \leq j \leq n$.

Then $G \simeq < x_1, ..., x_n \mid \Phi(r_1), ..., \Phi(r_{n-1}) >$.

In order to define a rewriting system $\Re'$ for G, we need to apply the symmetrization process on $\Phi(r_m)$, for $1 \leq m \leq n-1$ and to add of course the relations $x_i \overline{x_i} = 1, \overline{x_i} x_i = 1$ for $1 \leq i \leq n$.

We recall that each relator has the form: $r_m : x_i \overline{x_j} x_k \overline{x_l}$, so if none of the indices i, j, k, l is 0 then $\Phi(r_m) = x_i \overline{x_j} x_k \overline{x_l}$ and the symmetrization process applied on $\Phi(r_m)$ gives the set of relations $S(r_m)$ (see remark 2.1). If one of the indices is 0, assume $i = 0$, then $\Phi(r_m) = \overline{x_j} x_k \overline{x_l}$ and the symmetrization process applied on $\Phi(r_m)$ gives the following set of

relations, denoted by $S(\Phi(r_m))$:

$$(a)x_l = \overline{x_j} x_k \qquad (d)\overline{x_l} = \overline{x_k} x_j$$
$$(b)x_j = x_k \overline{x_l} \qquad (e)\overline{x_j} = x_l \overline{x_k}$$
$$(c)\overline{x_k} = \overline{x_l}\,\overline{x_j} \qquad (f)x_k = x_j x_l$$

We denote the relations $x_i \overline{x_i} = 1, \overline{x_i} x_i = 1$ for $1 \leq i \leq n$ by (0), and these relations will belong to all the sets of relations we will consider, even if we don't mention this explicitly. We will work all along with the assumption that $i = 0$, where i is the index of the first letter in $\Phi(r_m)$. Clearly, there is no loss of generality in doing this.

Now, let reverse the order of the operations on a relation r_m, i.e. first making the symmetrization process to obtain $S(r_m)$ and then applying

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Φ on both sides of the relations in $S(r_m)$. The set of relations obtained will be denoted by $\Phi(S(r_m))$. If none of the indices i, j, k, l in r_m is 0 then $\Phi(S(r_m)) = S(\Phi(r_m)) = S(r_m)$ and if $i = 0$ then $\Phi(S((r_m)))$ is the following set of relations:

$$\begin{matrix} x_l = \overline{x_j}x_k & \overline{x_l} = \overline{x_k}x_j \\ x_j = x_k\overline{x_l} & \overline{x_j} = x_l\overline{x_k} \end{matrix}$$

which correspond respectively to the relations $(a), (b), (d), (e)$ in $S(\Phi(r_m))$.

So, by reversing the order of the operations we lost the two relations $(c) : \overline{x_k} = \overline{x_l}\,\overline{x_j}$ and $(f) : x_k = x_j x_l$. Yet, we will show that the equivalence relation generated by $\Phi(S(r_m))$ is the same as that generated by $S(\Phi(r_m))$ and that in fact it is sufficient to consider an even smaller set of relations.

Claim 3.1. *The equivalence relation generated by the following set of*

$$(1) x_l = \overline{x_j}x_k$$

relations, denoted by Φ_m: $\quad (2) x_j = x_k\overline{x_l}$ *is the same as that generated*

$$(3) \overline{x_k} = \overline{x_l}\,\overline{x_j}$$

by $\Phi(S(r_m))$ and $S(\Phi(r_m))$.

Proof. First, the relations $(d), (e)$ and (f) can be derived from the relations $(1), (2), (3), (0)$ in the following way:

$(d) : \overline{x_k}x_j = \overline{x_k}(x_k\overline{x_l}) = \overline{x_l}$ by using (1) and then (0)

$(e) : x_l\overline{x_k} = (\overline{x_j}x_k)\overline{x_k} = \overline{x_j}$ by using (2) and then (0)

$(f) : x_j x_l = (x_k\overline{x_l})(\overline{x_j}x_k) = x_k(\overline{x_l}\,\overline{x_j})x_k = x_k(\overline{x_k})x_k = x_k$ by using $(1), (2), (3), (0)$.

So, the equivalence relation generated by $S(\Phi(r_m))$ is the same as that generated by Φ_m.

It remains to show that the relation (3) can be derived from the relations in $\Phi(S(r_m))$: (3) is obtained from (d) or (e) in one of the following ways:

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from (d): $\overline{x_l}\,\overline{x_j} = (\overline{x_k}x_j)\overline{x_j} = \overline{x_k}$ or from (e): $\overline{x_l}\,\overline{x_j} = \overline{x_l}(x_l\overline{x_k}) = \overline{x_k}$.
So, the equivalence relation generated by $\Phi(S(r_m))$ is the same as that
generated by Φ_m. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In order to define a rewriting system $\mathfrak{R}'$ for G, we have to orientate
the relations in Φ_m, for $1 \leq m \leq n-1$. Here, the complete rewriting
system $\mathfrak{R}$ found for H plays an important role: we will orientate the re-
lations in Φ_m in the same way as the corresponding relations in $S(r_m)$.
When x_0 does not appear in the relation r_m then things are clear since
$\Phi_m = S(r_m)$ but when x_0 appears in r_m then we have to recover for
each relation in Φ_m what is the corresponding relation in $S(r_m)$. It
seems to be a hard task but in fact it is not, since we know exactly how
rules look like in $\mathfrak{R}$ and these are described in remark 2.2. So, one can
see immediately that the relation (3) in Φ_m does not correspond to any
relation in $S(r_m)$ and one can guess that the relations corresponding
to (1) and (2) in $S(r_m)$ are respectively $\overline{x_0}x_l = \overline{x_j}x_k$ and $x_j\overline{x_0} = x_k\overline{x_l}$.
Since $x_0, \overline{x_0}$ are sources in the antipath $\hat{A}$ which defines $\mathfrak{R}$, we have
that $x_k, \overline{x_k}$ are also sources and $x_j, \overline{x_j}, x_l, \overline{x_l}$ are sinks in $\hat{A}$. So, in $\mathfrak{R}$
these relations are orientated $\overline{s_0}t_l \to \overline{t_j}s_k$ and $s_k\overline{t_l} \to t_j\overline{s_0}$ respectively
(see remark 2.2). Since we want to orientate the relations in Φ_m in
the same way as the corresponding relations in $S(r_m)$, we orientate the
relations (1) and (2) in the following way:
$(1): t_l \to \overline{t_j}s_k$
$(2): s_k\overline{t_l} \to t_j$
The following orientation is chosen for the relation (3) in Φ_m:
$(3): \overline{t_l}\,\overline{t_j} \to \overline{s_k}$
We will denote this set of 3 rules by $\vec{\Phi_m}$.

The rewriting system $\mathfrak{R}'$ for G is the union of the following sets of rules:

none of the indices 0	one of the indices 0	
$s_i\overline{t_j} \to t_l\overline{s_k}$	$(1): t_l \to \overline{t_j}s_k$	$(0): s_i\overline{s_i} \to 1$
$\overline{s_i}t_l \to \overline{t_j}s_k$	$(2): s_k\overline{t_l} \to t_j$	$(0): \overline{s_i}s_i \to 1$
$s_k\overline{t_l} \to t_j\overline{s_i}$	$(3): \overline{t_l}\,\overline{t_j} \to \overline{s_k}$	$(0): t_i\overline{t_i} \to 1$
$\overline{s_k}t_j \to \overline{t_l}s_i$		$(0): \overline{t_i}t_i \to 1$

Remark 3.2. *There is a rule in $\mathfrak{R}'$ whose left-hand side is a sink t with positive exponent if and only if this t is connected by an edge to $\overline{x_0}$ in $\hat{A}$, i.e it corresponds to a neighbouring region of the unbounded domain in the knot projection.*

Example: The rewriting system $\mathfrak{R}'$ for the eight knot group

$$t_2 \to \overline{t_1}s_4 \qquad t_3 \to \overline{t_2}s_5 \qquad t_1 \to \overline{t_3}s_5 \qquad s_4\overline{t_1} \to t_2\overline{s_5}$$

$$s_4\overline{t_2} \to t_1 \qquad s_5\overline{t_3} \to t_2 \qquad s_5\overline{t_1} \to t_3 \qquad \overline{s_4}t_2 \to \overline{t_1}s_5$$

$$\overline{t_2}\,\overline{t_1} \to \overline{s_4} \qquad \overline{t_3}\,\overline{t_2} \to \overline{s_5} \qquad \overline{t_1}\,\overline{t_3} \to \overline{s_5} \qquad s_5\overline{t_2} \to t_1\overline{s_4}$$

$$\overline{s_5}t_1 \to \overline{t_2}s_4$$

$$s_i\overline{s_i} \to 1 \qquad i = 0,4,5 \qquad \overline{s_i}s_i \to 1 \qquad i = 0,4,5$$

$$t_i\overline{t_i} \to 1 \qquad i = 1,2,3 \qquad \overline{t_i}t_i \to 1 \qquad i = 1,2,3$$

Note that $\mathfrak{R}'$ is not complete. As an example, the word $\overline{t_3}\,\overline{t_2}\,\overline{t_1}$ reduces to two different irreducible words $\overline{s_5}\,\overline{t_1}$ and $\overline{t_3}\,\overline{s_4}$.

3.2. **Properties of $\mathfrak{R}'$.**

Claim 3.3. *If there is a relator $r_m : x_i\overline{x_j}x_k\overline{x_l}$ in the augmented Dehn presentation of an alternating knot group, then there are $1 \le k', l' \le n$ such that there is also the relator $r_{m'} : x_{k'}\overline{x_j}x_i\overline{x_{l'}}$.*

Proof. Since the knot is alternating and we have $r_m : x_i\overline{x_j}x_k\overline{x_l}$, then we have necessarily in the knot projection the situation described in

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Fig.4. This means that there is also the relation $r_{m'} : x_{k'}\overline{x_j}x_i\overline{x_{l'}}$ (or its cyclic permutation $r_{m'} : x_i\overline{x_{l'}}x_{k'}\overline{x_j}$) $\qquad\square$

Claim 3.4. *If the set* $\vec{\Phi}_m$:
$$
\begin{aligned}
(1_{l,j,k}) &: t_l \to \overline{t_j}s_k \\
(2_{l,j,k}) &: s_k\overline{t_l} \to t_j \quad \text{is in } \mathfrak{R}' \text{ then} \\
(3_{l,j,k}) &: \overline{t_l}\,\overline{t_j} \to \overline{s_k}
\end{aligned}
$$

$$
\vec{\Phi}_{m'} : \quad
\begin{aligned}
(1_{j,l',k'}) &: t_j \to \overline{t_{l'}}s_{k'} \\
(2_{j,l',k'}) &: s_{k'}\overline{t_j} \to t_{l'} \quad \text{is also in } \mathfrak{R}'. \\
(3_{j,l',k'}) &: \overline{t_j}\,\overline{t_{l'}} \to \overline{s_{k'}}
\end{aligned}
$$

Proof. It follows from claim 3.3 , where $\vec{\Phi}_m$ and $\vec{\Phi}_{m'}$ are the sets of rules in $\mathfrak{R}'$ corresponding to r_m and $r_{m'}$ respectively. $\qquad\square$

3.3. **Definition of an ordering on the words in** $(X \cup \overline{X})^*$**.** By abuse of notation, we denote also by $X \cup \overline{X}$ the union of the set of sinks and the set of sources, i.e the set of generators and their inverses after their renaming. We write a non-empty word w in $(X \cup \overline{X})^*$, $w = S_1 t_1 S_2 t_2 .. S_k t_k S_{k+1}$, with S_i a sequence of sources $s_1 s_2 .. s_{n_i}$ such that the length of S_i (i.e the number of occurrences of s), denoted by n_i, satisfies $0 \le n_i \le \ell(w)$ for $1 \le i \le k+1$.

We will define an ordering on the words in $(X \cup \overline{X})^*$ in the following way. Let $>$ denote the usual ordering on the nonnegative integers and let $>_{lex}$ denote the lexicographic ordering from the left on $4-$tuples of nonnegative integers, induced by $>$. Define

$V_1(w)$ to be the number of occurrences of sinks with positive and negative exponent (the number of t) in w. It holds that $V_1(w) \le \ell(w)$.

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$V_2(w)$ to be the number of occurrences of sinks with positive exponent in w, but only those sinks which are connected by an edge to $\overline{x_0}$ in the antipath $\hat{A}$ which defines $\mathfrak{R}$. These generators correspond in fact to neighbouring regions of the unbounded region (x_0) in the knot projection. It holds that $V_2(w) \leq \ell(w)$.

$V_3(w) = \sum_{j=1}^{j=k} \sum_{i=0}^{i=k-j} 2^i n_j$

It holds that $V_3(w) \leq 2^{2k}\ell(w)$.

$V_4(w)$ to be $\ell(w)$, the length of w.

$V(w) = (V_1(w), V_2(w), V_3(w), V_4(w))$

Observe that $V(w) \geq (0,0,0,0)$ for every non-empty word w. Define $w > w'$ if and only if $V(w) >_{lex} V(w')$

Example: Computation of $V(t_2)$ and $V(\overline{t_1}s_4)$ It holds that $V_1(t_2) = V_1(\overline{t_1}s_4) = 1$ but $V_2(t_2) = 1$ and $V_2(\overline{t_1}s_4) = 0$. So, $V(t_2) >_{lex} V(\overline{t_1}s_4)$.

4. Definition of an equivalent rewriting system $\mathfrak{R}''$

Lemma 4.1. *Assume* $\vec{\Phi}_m$:
$$(1_{l,j,k}) : t_l \to \overline{t_j}s_k \qquad\qquad (1_{j,l',k'}) : t_j \to \overline{t_{l'}}s_{k'}$$
$$(2_{l,j,k}) : s_k\overline{t_l} \to t_j \quad and \ \vec{\Phi}_{m'} : \quad (2_{j,l',k'}) : s_{k'}\overline{t_j} \to t_{l'}$$
$$(3_{l,j,k}) : \overline{t_l}\,\overline{t_j} \to \overline{s_k} \qquad\qquad (3_{j,l',k'}) : \overline{t_j}\,\overline{t_{l'}} \to \overline{s_{k'}}$$
are in $\mathfrak{R}'$. *Then the rule* $(4)\overline{s_k}\,\overline{t_{l'}} \to \overline{t_l}\,\overline{s_{k'}}$ *is obtained from the equivalence relation generated by rules of kind* (3).

Proof. We have $\overline{s_k}\,\overline{t_{l'}} = \overline{t_l}\,\overline{t_j}\,\overline{t_{l'}} = \overline{t_l}\,\overline{s_{k'}}$, using first the equivalence relation generated by $(3_{l,j,k})$ and then by $(3_{j,l',k'})$. In order to orientate

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this relation, we use the ordering defined above: $V(\overline{s_k}\,\overline{t_{l'}}) = (1,0,1,2)$ and $V(\overline{t_l}\,\overline{s_{k'}}) = (1,0,0,2)$. So we have $\overline{s_k}\,\overline{t_{l'}} \to \overline{t_l}\,\overline{s_{k'}}$. $\square$

Remark 4.2. *Adding the rule* $(4)\overline{s_k}\,\overline{t_{l'}} \to \overline{t_l}\,\overline{s_{k'}}$ *to* $\Re'$ *will resolve the critical pairs resulting from the overlapping of the rules* $(3_{l,j,k})$ *and* $(3_{j,l',k'})$.

Example: In the rewriting system $\Re'$ for the figure-eight knot group we have the following two rules of kind (3): $\overline{x_2}\,\overline{x_1} \to \overline{x_4}$ and $\overline{x_1}\,\overline{x_3} \to \overline{x_5}$. The rule of kind (4) obtained from the equivalence relation generated by these two rules is then: $\overline{x_4}\,\overline{x_3} \to \overline{x_2}\,\overline{x_5}$.

Lemma 4.3. *Assume* $\vec{\Phi_m}$:
$$\begin{array}{ll} (1_{l,j,k}) : t_l \to \overline{t_j}s_k & (\alpha)s_m\overline{t_p} \to t_l\overline{s_{m'}} \\ (2_{l,j,k}) : s_k\overline{t_l} \to t_j \quad and & (\beta)s_{m'}\overline{t_l} \to t_p\overline{s_m} \\ (3_{l,j,k}) : \overline{t_l}\,\overline{t_j} \to \overline{s_k} & (\gamma)\overline{s_m}t_l \to \overline{t_p}s_{m'} \\ & (\delta)\overline{s_{m'}}t_p \to \overline{t_l}s_m \end{array}$$
are in $\Re'$. *Then the rule* $(5)\overline{s_m}\,\overline{t_j} \to \overline{t_p}s_{m'}\,\overline{s_k}$ *is obtained from the equivalence relation generated by* $(3_{l,j,k})$ *and* (β). *Further, rule* (γ) *can be derived using the rule* (5).

Proof. We have $\overline{t_p}s_{m'}\,\overline{s_k} = \overline{t_p}s_{m'}\overline{t_l}\,\overline{t_j} = \overline{t_p}t_p\overline{s_m}\,\overline{t_j} = \overline{s_m}\,\overline{t_j}$, using first the equivalence relation generated by $(3_{l,j,k})$ and then by (β) and (0). In order to orientate this relation, we use the ordering defined above: $V(\overline{s_m}\,\overline{t_j}) = (1,0,1,2)$ and $V(\overline{t_p}s_{m'}\,\overline{s_k}) = (1,0,0,3)$. So, we have $\overline{s_m}\,\overline{t_j} \to \overline{t_p}s_{m'}\,\overline{s_k}$. Further, we have $\overline{s_m}t_l \to \overline{s_m}\,\overline{t_j}s_k \to \overline{t_p}s_{m'}\overline{s_k}s_k \to \overline{t_p}s_{m'}$ by using rules $(1_{l,j,k})$, (5) and (0). $\square$

Remark 4.4. *Adding the rule* $(5)\overline{s_m}\,\overline{t_j} \to \overline{t_p}s_{m'}\,\overline{s_k}$ *to* $\Re'$ *will resolve the critical pairs resulting from the overlapping of the rules* $(3_{l,j,k})$ *and* β *and from the inclusion ambiguity of rule* $(1_{l,j,k})$ *and* γ. *Additionally,*

removing the rule γ from $\Re'$ and replacing it by (5) implies that t_l cannot appear in the left-hand side of any rule except in $(1_{l,j,k})$.

Example: In the rewriting system $\Re'$ for the figure-eight knot group we have the two following rules of kind (3) and (β) respectively: $\overline{x_2}\,\overline{x_1} \to \overline{x_4}$ and $x_5\overline{x_2} \to x_1\overline{x_4}$. The rule of kind (5) obtained from the equivalence relation generated by these two rules is then: $\overline{x_4}\,\overline{x_1} \to \overline{x_1}x_5\overline{x_4}$.

Proposition 4.5. *Assume that $\Re'$ is the union of the following sets of rules:*

	$\vec{\phi_m}$		$\vec{\phi_{m'}}$	
$(\alpha)s_m\overline{t_p} \to t_l\overline{s_{m'}}$	$(1_{l,j,k}): t_l \to \overline{t_j}s_k$	$(1_{j,l',k'}): t_j \to \overline{t_{l'}}s_{k'}$
$(\beta)s_{m'}\overline{t_l} \to t_p\overline{s_m}$	$(2_{l,j,k}): s_k\overline{t_l} \to t_j$	$(2_{j,l',k'}): s_{k'}\overline{t_j} \to t_{l'}$
$(\gamma)\overline{s_m}t_l \to \overline{t_p}s_{m'}$	$(3_{l,j,k}): \overline{t_l}\,\overline{t_j} \to \overline{s_k}$	$(3_{j,l',k'}): \overline{t_j}\,\overline{t_{l'}} \to \overline{s_{k'}}$
$(\delta)\overline{s_{m'}}t_p \to \overline{t_l}s_m$		

Let $\Re''$ be the rewriting system obtained by adding to $\Re'$ the rules $(4)\overline{s_k}\,\overline{t_{l'}} \to \overline{t_l}\,\overline{s_{k'}}$ and $(5)\overline{s_m}\,\overline{t_j} \to \overline{t_p}s_{m'}\,\overline{s_k}$, replacing (α) by $(\alpha'): s_m\overline{t_p} \to \overline{t_j}s_k\overline{s_{m'}}$ and $(2_{j,l,k})$ by $(2'_{j,l,k}): s_k\overline{t_l} \to \overline{t_{l'}}s_{k'}$ and removing (γ). That means that $\Re''$ is the union of the following sets of rules:

	$\vec{\phi_m}$		$\vec{\phi_{m'}}$	
$(\alpha')s_m\overline{t_p} \to \overline{t_j}s_k\overline{s_{m'}}$	$(1_{l,j,k}): t_l \to \overline{t_j}s_k$	$(1_{j,l',k'}): t_j \to \overline{t_{l'}}s_{k'}$
$(\beta)s_{m'}\overline{t_l} \to t_p\overline{s_m}$	$(2'_{l,j,k}): s_k\overline{t_l} \to \overline{t_{l'}}s_{k'}$	$(2_{j,l',k'}): s_{k'}\overline{t_j} \to t_{l'}$
$(\delta)\overline{s_{m'}}t_p \to \overline{t_l}s_m$	$(3_{l,j,k}): \overline{t_l}\,\overline{t_j} \to \overline{s_k}$	$(3_{j,l',k'}): \overline{t_j}\,\overline{t_{l'}} \to \overline{s_{k'}}$
$(5): \overline{s_m}\,\overline{t_j} \to \overline{t_p}s_{m'}\,\overline{s_k}$	$(4): \overline{s_k}\,\overline{t_{l'}} \to \overline{t_l}\,\overline{s_{k'}}$	

Then the rewriting system $\Re''$ is equivalent to $\Re'$ and is reduced.

Proof. From lemmas 4.1 and 4.3, the rules (4) and (5) are obtained from the equivalence relation generated by the rules in $\Re'$. (α') and $(2'_{j,l,k})$ are obtained by reducing the right-hand side of (α) and $(2_{j,l,k})$ using

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the rule $(1_{l,j,k})$ and (γ) can be derived using (5). So, $\Re''$ is equivalent to $\Re'$. $\Re''$ is reduced since the right-hand sides of the rules in $\Re''$ have been reduced and the rules (of kind γ) for which the left-hand side contains a subword which is the left-hand side of an other rule have been removed. $\qquad\square$

Remark 4.6. *If additionally there is a rule of kind (1) which has t_p as left-hand side then the rule (δ) is replaced by a rule of kind (5) and we replace (β) by (β') in which the right-hand side is reduced.*

Example: The rewriting system $\Re''$ for the figure-eight knot group

$$t_2 \to \overline{t_1}s_4 \qquad t_3 \to \overline{t_2}s_5 \qquad t_1 \to \overline{t_3}s_5 \qquad s_4\overline{t_1} \to \overline{t_1}s_4\overline{s_5}$$

$$s_4\overline{t_2} \to \overline{t_3}s_5 \qquad s_5\overline{t_3} \to \overline{t_1}s_4 \qquad s_5\overline{t_1} \to \overline{t_2}s_5 \qquad \overline{s_4}\overline{t_1} \to \overline{t_1}s_5\overline{s_4}$$

$$\overline{t_2}\,\overline{t_1} \to \overline{s_4} \qquad \overline{t_3}\,\overline{t_2} \to \overline{s_5} \qquad \overline{t_1}\,\overline{t_3} \to \overline{s_5} \qquad s_5\overline{t_2} \to \overline{t_3}s_5\overline{s_4}$$

$$\overline{s_4}\,\overline{t_3} \to \overline{t_2}\,\overline{s_5} \qquad \overline{s_5}\,\overline{t_2} \to \overline{t_1}\,\overline{s_5} \qquad \overline{s_5}\,\overline{t_1} \to \overline{t_3}\,\overline{s_4} \qquad \overline{s_5}\,\overline{t_3} \to \overline{t_2}s_4\overline{s_5}$$

$$s_i\overline{s_i} \to 1 \qquad i = 0, 4, 5 \qquad \overline{s_i}s_i \to 1 \qquad i = 0, 4, 5$$

$$t_i\overline{t_i} \to 1 \qquad i = 1, 2, 3 \qquad \overline{t_i}t_i \to 1 \qquad i = 1, 2, 3$$

Note that $\overline{s_5}t_1 \to \overline{t_2}s_4$ and $\overline{s_4}t_2 \to \overline{t_1}s_5$ (in the fourth set of rules) have been replaced by the rules of kind (5), $\overline{s_5}\,\overline{t_3} \to \overline{t_2}s_4\overline{s_5}$ and $\overline{s_4}\,\overline{t_1} \to \overline{t_1}s_5\overline{s_4}$ respectively. The rewriting system $\Re''$ for the trefoil knot group is:
$\Re'' = \{x_1 \to \overline{x_4}x_3, x_3\overline{x_1} \to \overline{x_2}x_3, \overline{x_1}\,\overline{x_4} \to \overline{x_3}\} \cup \{x_2 \to \overline{x_1}x_3, x_3\overline{x_2} \to \overline{x_4}x_3, \overline{x_2}\,\overline{x_1} \to \overline{x_3}\} \cup \{x_4 \to \overline{x_2}x_3, x_3\overline{x_4} \to \overline{x_1}x_3, \overline{x_4}\,\overline{x_2} \to \overline{x_3}\} \cup \{\overline{x_3}\,\overline{x_2} \to \overline{x_1}\,\overline{x_3}, \overline{x_3}\,\overline{x_1} \to \overline{x_4}\,\overline{x_3}, \overline{x_3}\,\overline{x_4} \to \overline{x_2}\,\overline{x_3}\} \cup \{x_i\overline{x_i} \to 1, \overline{x_i}x_i \to 1, for 0 \le i \le 4\}$ (see [2] for notation) Note that for the trefoil knot there are no rules of kind $\alpha, \beta, \gamma, \delta$ and (5) since all the regions have an edge in common with the unbounded region x_0.

The rewriting system $\Re''$ is complete and this is what we will show in a more general context in the following.

5. COMPLETENESS OF THE REWRITING SYSTEM $\Re''$

5.1. **Termination of the rewriting system $\Re''$.** As before, we will use the following notation: t denotes a sink in the antipath $\hat{A}$ with positive or negative exponent, t^+ denotes a sink with positive exponent which is connected by an edge to $\overline{x_0}$ in the antipath $\hat{A}$. s denotes a source in $\hat{A}$ with positive or negative exponent and S_i a sequence $s_1 s_2 .. s_{n_i}$.

We will show that we can divide schematically all the rules in $\Re''$ (except rules of kind (0)) in the following four classes:

$$(A) \quad st \to tss$$
$$(B) \quad st \to ts$$
$$(C) \quad t^+ \to ts$$
$$(D) \quad tt \to s$$

Note that the s, t occurring in one side of a relation are different from the s, t in the other side, although we use the same letters.

Lemma 5.1. *Each rule in $\Re''$ belongs to one of the classes (A),(B),(C) or (D) described above.*

Proof. The rules of kind $(5), \alpha'$ and β' belong to the class (A). The rules of kind $(2'), (4)$ and $\alpha, \beta, \gamma, \delta$ belong to class (B). The rules of kind (1) belong to class (C)and the rules of kind (3) belong to class (D). $\quad\square$

Lemma 5.2. *In $\Re''$, if a specific t^+ occurs in a rule from class (C), then the same t^+ cannot occur in any other rule, except in rules of kind (0).*

Proof. From proposition 4.5, we have that $\Re''$ is reduced, so a t^+ which occurs in a rule from class (C) cannot occur in the right-hand side of any rule. From the description of the rules in $\Re'$ and $\Re''$ in proposition 4.5, we have that a t with positive exponent sign can occur only in rules of kind (1), γ, δ and (0), i.e in rules from class (C) or (B) respectively. Now, if there is a t with positive exponent sign, denoted by t_l for $1 \leq l \leq n$, which occurs both in a rule of kind (1) and in a rule of kind (γ), then in the construction of $\Re''$ the rule of kind (γ) has been replaced by a rule of kind (5) in which there is no t with positive exponent sign. The same argument holds if if there is a t with positive exponent sign which occurs both in a rule of kind (1) and in a rule of kind (δ), since (γ) and (δ) have the same form (see proposition 4.5). $\qquad \square$

Lemma 5.3. *Let $w = S_1 t_1 S_2 t_2 .. S_k t_k$ be a non-empty word in $(X \cup \overline{X})^*$, with S_i a sequence of sources such that the length of S_i, $\ell(S_i)$, satisfies $0 \leq \ell(S_i) \leq \ell(w)$ for $1 \leq i \leq k$. Then the application of any rule in $\Re''$ on w leads to one of the following three cases for the "t" which participated in the left-hand side of the rule:*

1. t moves one step to the left and there is at most one source added to w.

2. t stays at the same place and there is one source added to w.

3. t disappears and there is at most one source added to w. In fact, each application of a rule from class (D) or of kind $t\overline{t} \to 1$ or $\overline{t}t \to 1$ on w reduces the number of "t" by 2.

Proof. The first case occurs when a rule from class (A) or (B) is applied, the second case when a rule from class (C) and the last case when a rule from class (D) or a rule of kind (0) is applied. $\qquad \square$

Theorem 5.4. *The rewriting system $\Re''$ is terminating.*

Proof. Let $w = S_1 t_1 S_2 t_2 .. S_k t_k S_{k+1}$ be a word in $(X \cup \overline{X})^*$, with $0 \leq n_i \leq \ell(w)$ for $1 \leq i \leq k+1$. We will show that after the application of any rule from $\Re''$ on w, we obtain a word w' such that $V(w') <_{lex} V(w)$. We denote $w' = S_1' t_1' S_2' t_2' .. S_k' t_k' S_{k+1}'$ where $\ell(S_i') = n_i'$ for $1 \leq i \leq k+1$.

If a rule from class (D) is applied on w, then $V_1(w') = V_1(w) - 2$ since each single application of a rule from class (D) on a word w reduces the number of "t" by 2. This implies that $V(w') <_{lex} V(w)$. The same holds for all the rules of kind (0): $t\overline{t} \to 1$ and $\overline{t}t \to 1$.

If a rule from class (C) is applied on w, then $V_1(w) = V_1(w')$ since the number of occurrences of t does not change but $V_2(w') = V_2(w) - 1$. This is due to the fact that the t^+ on which the rule is applied is counted in $V_2(w)$ (see remark 3.2). So, $V(w') <_{lex} V(w)$.

From lemma 5.3, each application of a rule from class $(A), (B)$ on w keeps the number of "t" the same, so $V_1(w) = V_1(w')$. Moreover, the "t^+" which are counted in $V_2(w)$ cannot appear in the left-hand side, nor in the right-hand side of any rule from class $(A), (B)$, so $V_2(w) = V_2(w')$.

First, assume a rule from class (A) is applied on w and let denote by t_p $(1 \leq p \leq k)$ the t on which the rule is applied. So, it holds that $n_i' = n_i$ for $i \neq p, p+1$, $n_p' = n_p - 1$ and $n_{p+1}' = n_{p+1} + 2$. The computation of $V_3(w')$ gives the following:

$V_3(w') = \sum_{j=1}^{j=k} \sum_{i=0}^{i=k-j} 2^i n_j' = \sum_{j=1}^{j=p-1} \sum_{i=0}^{i=k-j} 2^i n_j' + \sum_{i=0}^{i=k-p} 2^i n_p' + \sum_{i=0}^{i=k-p-1} 2^i n_{p+1}' + \sum_{j=p+2}^{j=k} \sum_{i=0}^{i=k-j} 2^i n_j' = \sum_{j=1}^{j=p-1} \sum_{i=0}^{i=k-j} 2^i n_j + \sum_{i=0}^{i=k-p} 2^i (n_p -$

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$1)+\sum_{i=0}^{i=k-p-1} 2^i(n_{p+1}+2)+\sum_{j=p+2}^{j=k}\sum_{i=0}^{i=k-j} 2^i n_j = V_3(w)-1(\sum_{i=0}^{i=k-p} 2^i)+$

$2(\sum_{i=0}^{i=k-p-1} 2^i) = V_3(w)-2^{k-p}+1(\sum_{i=0}^{i=k-p-1} 2^i) = V_3(w)-2^{k-p}+(2^{k-p}-$

$1)/(2-1) = V_3(w)-1$

So, $V_3(w') < V_3(w)$, which implies that $V(w') <_{lex} V(w)$.

Next, assume a rule from class (B) is applied on w and let denote by t_p $(1 \le p \le k)$ the t on which the rule is applied. So, it holds that $n_i' = n_i$ for $i \ne p, p+1$, $n_p' = n_p - 1$ and $n_{p+1}' = n_{p+1} + 1$. The computation of $V_3(w')$ gives the following:

$V_3(w') = \sum_{j=1}^{j=k}\sum_{i=0}^{i=k-j} 2^i n_j' = V_3(w)-1(\sum_{i=0}^{i=k-p} 2^i)+1(\sum_{i=0}^{i=k-p-1} 2^i) = V_3(w) - 2^{k-p}$

So, $V_3(w') < V_3(w)$, which implies that $V(w') <_{lex} V(w)$.

At last, if a rule of kind $s\bar{s} \to 1$ or $\bar{s}s \to 1$ is applied on w, then $V_1(w') = V_1(w)$, $V_2(w') = V_2(w)$, since the number of occurrences of t of any kind does not change. In order to compare $V_3(w)$ and $V_3(w')$, we have to check several cases.

Assume there is at least one occurrence of t in w. If there is no occurrence of t at the right of the subword on which a rule of kind $s\bar{s} \to 1$ or $\bar{s}s \to 1$ is applied, i.e $s\bar{s}$ or $\bar{s}s$ is a subword of S_{k+1}, then $V_3(w') = V_3(w)$, since the computation of $V_3(w)$ does not take into account n_{k+1}, the length of S_{k+1}. At last, $V_4(w') = V_4(w) - 2$, which implies $V(w') <_{lex} V(w)$.

Otherwise, if there is an occurrence of t, denoted by t_p, at the right of the subword on which a rule of kind $s\bar{s} \to 1$ or $\bar{s}s \to 1$ is applied, then $V_3(w')$ is not the same as $V_3(w)$, since $n'_p = n_p - 2$ and $p \neq k+1$. The computation of $V_3(w')$ gives the following:

$V_3(w') = \sum_{j=1}^{j=k} \sum_{i=0}^{i=k-j} 2^i n'_j = V_3(w) - 2(\sum_{i=0}^{i=k-p} 2^i)$

So, $V_3(w') < V_3(w)$, which implies that $V(w') <_{lex} V(w)$.

Assume there is no occurrence of t in w, then $w = S_1$. Since the application of a rule of kind $s\bar{s} \to 1$ or $\bar{s}s \to 1$ reduces the length of S_1 by 2, we have that $n'_1 = n_1 - 2$. The computation of $V_3(w')$ gives the following:

$V_3(w') = \sum_{j=1}^{j=1} \sum_{i=0}^{i=1-j} 2^i n'_j = n'_1 = n_1 - 2 = V_3(w) - 2$

So, $V(w') <_{lex} V(w)$.

So, we have that after the application of any rule from $\Re''$ on w, we obtain a word w' which satisfies $V(w') <_{lex} V(w)$ and this implies that $\Re''$ is terminating . $\qquad\square$

5.2. **The main result: Completeness of the rewriting system $\Re''$.**

Theorem 5.5. *The rewriting system $\Re''$ is finite and complete.*

Proof. $\Re''$ is terminating so it remains to show that $\Re''$ is locally confluent, i.e that all the critical pairs, obtained from overlappings between rules and from inclusion ambiguities of some rules in other rules, resolve with no addition of new rules. This is done in the lemmas in the next section, since the arguments used there are mostly technical. $\quad\square$

The sets of rules in $\mathfrak{R}''$ are described in the lemmas 4.1 and 4.3. In the following table we will give the list of critical pairs we have to check, where "-" means that there is nothing to check and "$\sqrt{}$" means that there is an overlapping/inclusion which is checked. A "$\sqrt{}$" at row i column j means that there is an overlapping of a rule of kind i with a rule of kind j, where the rule of kind i is at left and the rule of kind j is at right. As an example, inclusion ambiguities occur only between rules of kind (1) and rules of kind (0). There can be no overlapping between the rules of kind (2'), (4), (5), α', β and δ, since in their left-hand side the first letter is some "s" and the last letter is some "t".

	0	1	2'	3	4	5	α'	β	δ
0	-	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$	$\sqrt{}$
1	$\sqrt{}$	-	-	-	-	-	-	-	-
2'	$\sqrt{}$	-	-	$\sqrt{}$	-	-	-	-	-
3	$\sqrt{}$	-	-	$\sqrt{}$	-	-	-	-	-
4	$\sqrt{}$	-	-	$\sqrt{}$	-	-	-	-	-
5	$\sqrt{}$	-	-	$\sqrt{}$	-	-	-	-	-
α'	$\sqrt{}$	-	-	-	-	-	-	-	-
β	$\sqrt{}$	-	-	$\sqrt{}$	-	-	-	-	-
δ	$\sqrt{}$	-	-	-	-	-	-	-	-

5.3. Locally confluence of $\mathfrak{R}''$.

Lemma 5.6. *Critical pairs resulting from inclusion ambiguity of rule (1) and rule (0) resolve.*

$$t_l\overline{t_l} \qquad and \qquad \overline{t_l}t_l$$

$$(1) \quad \swarrow \quad \searrow \quad (0) \qquad (1) \quad \swarrow \quad \searrow$$

$$\overline{t_j}s_k\overline{t_l} \qquad\qquad 1 \qquad \overline{t_l}\,\overline{t_j}s_k \qquad\qquad 1$$

Proof. $\downarrow \quad (2') \qquad\qquad\qquad \downarrow \quad (3)$ □

$$\overline{t_j}\,\overline{t_{l'}}s_{k'} \qquad\qquad\qquad \overline{s_{k'}}s_{k'} \quad \hookrightarrow$$

$$\downarrow \quad (3)$$

$$\overline{s_{k'}}s_{k'} \quad \hookrightarrow$$

Lemma 5.7. *Critical pairs resulting from overlapping between rule* (0) *and rule* (2') *resolve.*

$$s_k\overline{t_l}t_l \qquad and \qquad \overline{s_k}s_k\overline{t_l}$$

$$(2') \quad \swarrow \quad \searrow \quad (0) \qquad (2') \quad \swarrow \quad \searrow$$

$$\overline{t_{l'}}s_{k'}t_l \qquad\qquad s_k \qquad \overline{s_k}\,\overline{t_{l'}}s_{k'} \qquad\qquad \overline{t_l}$$

Proof. $\downarrow \quad (1) \qquad\qquad\qquad\qquad \downarrow \quad (4)$ □

$$\overline{t_{l'}}s_{k'}\overline{t_j}s_k \qquad\qquad\qquad \overline{t_l}\,\overline{s_{k'}}s_{k'} \quad \hookrightarrow$$

$$\downarrow \quad (2)$$

$$\overline{t_{l'}}t_{l'}s_k \quad \hookrightarrow$$

Lemma 5.8. *Critical pairs resulting from overlapping between rule* (0) *and rule* (3) *resolve.*

$$t_l\overline{t_l}\,\overline{t_j} \qquad and \qquad \overline{t_l}\,\overline{t_j}t_j$$

$$(3) \quad \swarrow \quad \searrow \quad (0) \qquad (3) \quad \swarrow \quad \searrow$$

$$t_l\overline{s_k} \qquad\qquad \overline{t_j} \qquad \overline{s_k}t_j \qquad\qquad \overline{t_l}$$

Proof. $\downarrow \quad (1) \qquad\qquad\qquad\qquad \downarrow \quad (1)$ □

$$\overline{t_j}s_{k'}\overline{s_k} \qquad\qquad\qquad \overline{s_k}\,\overline{t_{l'}}s_{k'}$$

$$\downarrow \quad (0) \qquad\qquad\qquad\qquad \downarrow \quad (4)$$

$$\overline{t_j} \quad \hookrightarrow \qquad\qquad \overline{t_l}\,\overline{s_{k'}}s_{k'} \quad \hookrightarrow$$

Lemma 5.9. *Critical pairs resulting from overlapping between rule* (0) *and rule* (4) *resolve.*

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$$s_k\overline{s_k}\,\overline{t_{l'}} \qquad and \qquad \overline{s_k}\,\overline{t_{l'}}t_{l'}$$

$$(4) \quad \swarrow \qquad \searrow \quad (0) \qquad\qquad (4) \quad \swarrow \qquad \searrow$$

$$s_k\overline{t_l}\,\overline{s_{k'}} \qquad\qquad \overline{t_{l'}} \qquad\qquad \overline{t_l}\,\overline{s_{k'}}\,t_{l'} \qquad\qquad \overline{s_k}$$

Proof. $\qquad \downarrow \quad (2') \qquad\qquad\qquad\qquad\qquad \downarrow \qquad (1) \qquad\qquad \square$

$$\overline{t_{l'}}s_{k'}\overline{s_{k'}} \quad \hookrightarrow \qquad\qquad\qquad\qquad \overline{t_l}\,\overline{s_{k'}}\,\overline{t_q}s_{q'}$$

$$(0) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \qquad (4)$$

$$\overline{t_l}\,\overline{t_j}\,\overline{s_{q'}}s_{q'} \qquad \hookrightarrow \qquad (3)$$

Lemma 5.10. *Critical pairs resulting from overlapping between rule (0) and rule (5) resolve.*

$$s_m\overline{s_m}\overline{t_j} \qquad and \qquad \overline{s_m}\overline{t_j}t_j$$

$$(5) \quad \swarrow \qquad \searrow \quad (0) \qquad\qquad (5) \quad \swarrow \qquad \searrow$$

$$s_m\overline{t_p}s_{m'}\overline{s_k} \qquad\qquad \overline{t_j} \qquad\qquad \overline{t_p}\,s_{m'}\overline{s_k}t_j \qquad\qquad \overline{s_m}$$

$$\downarrow \quad (\alpha') \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \qquad (1)$$

Proof. $\;\; \overline{t_j}s_k\overline{s_{m'}}s_{m'}\overline{s_k} \quad \hookrightarrow \qquad\qquad\qquad \overline{t_p}s_{m'}\overline{s_k}\,\overline{t_{l'}}s_{k'}$

$$\downarrow \qquad (4)$$

$$\overline{t_p}s_{m'}\overline{t_l}\,\overline{s_{k'}}s_{k'}$$

$$\downarrow \qquad (0),(3)$$

$$\overline{t_p}t_p\overline{s_m} \hookrightarrow$$

$$\square$$

Lemma 5.11. *Critical pairs resulting from overlapping between rule (0) and rule (α') resolve.*

$$s_m\overline{t_p}\,t_p \qquad and \qquad \overline{s_m}s_m\overline{t_p}$$

$$(\alpha') \quad \swarrow \qquad \searrow \quad (0) \qquad\qquad (\alpha') \quad \swarrow \qquad \searrow$$

$$\overline{t_j}s_k\overline{s_{m'}}t_p \qquad\qquad\qquad s_m \qquad\qquad \overline{s_m}\,\overline{t_j}s_k\overline{s_{m'}} \qquad\qquad\qquad \overline{t_p}$$

$$\downarrow \quad (\delta) \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \qquad (5)$$

Proof. $\quad \overline{t_j}s_k\overline{t_l}s_m \qquad\qquad\qquad\qquad\qquad\qquad \overline{t_p}s_{m'}\overline{s_k}s_k\overline{s_{m'}} \qquad \hookrightarrow$

$$\downarrow \quad (2')$$

$$\overline{t_j}\,\overline{t_{l'}}s_{k'}s_m$$

$$\downarrow \quad (3)$$

$$\overline{s_{k'}}s_{k'}s_m \qquad \hookrightarrow$$

$$\square$$

Lemma 5.12. *Critical pairs resulting from overlapping between rule (0) and rule (β) resolve.*

$$s_{m'}\overline{t_l}\,t_l \qquad and \qquad \overline{s_{m'}}s_{m'}\overline{t_l}$$

$$(\beta) \quad \swarrow \qquad \searrow \quad (0) \qquad\qquad (\beta) \quad \swarrow \qquad \searrow \quad (0)$$

$$t_p\overline{s_m}t_l \qquad\qquad\qquad s_{m'} \qquad\qquad \overline{s_{m'}}t_p\overline{s_m} \qquad\qquad\qquad \overline{t_l}$$

Proof. $\qquad \downarrow \quad (1) \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \qquad (\delta)$

$$t_p\overline{s_m}\,\overline{t_j}s_k \qquad\qquad\qquad\qquad\qquad\qquad \overline{t_l}s_m\overline{s_m} \qquad\qquad\qquad \hookrightarrow$$

$$\downarrow \quad (5)$$

$$\overline{t_p}\,\overline{t_p}s_{m'}\overline{s_k}s_k \qquad\qquad \hookrightarrow$$

$$\square$$

Lemma 5.13. *Critical pairs resulting from overlapping between rule (0) and rule (δ) resolve.*

$$s_{m'}\overline{s_{m'}}t_p \qquad \text{and} \qquad \overline{s_{m'}}t_p\overline{t_p}$$

$$\quad (\delta) \quad \swarrow \quad \searrow \quad (0) \qquad (\delta) \quad \swarrow \quad \searrow$$

$$s_{m'}\overline{t_l}s_m \qquad\qquad t_p \qquad \overline{t_l}\,s_m\overline{t_p} \qquad\qquad \overline{s_{m'}}$$

Proof. $\qquad \downarrow \quad (\beta) \qquad\qquad\qquad\qquad \downarrow \quad (\alpha') \qquad\qquad \square$

$$t_p\overline{s_m}s_m \quad \hookrightarrow \qquad\qquad\qquad \overline{t_l}\,\overline{t_j}s_k\overline{s_{m'}}$$

$$\downarrow \qquad (3)$$

$$\overline{s_k}\,s_k\overline{s_{m'}} \qquad\qquad \hookrightarrow$$

Lemma 5.14. *Critical pairs resulting from overlapping between rule $(2')$ and rule (3) resolve.*

$$s_k\overline{t_l}\,\overline{t_j}$$

$$(2') \quad \swarrow \qquad\qquad \searrow \quad (3)$$

$$\overline{t_{l'}}s_{k'}\overline{t_j} \qquad\qquad\qquad s_k\overline{s_k}$$

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

$$\downarrow \quad (2) \qquad\qquad \downarrow$$

$$\overline{t_{l'}}t_{l'} \quad \hookrightarrow \qquad\qquad\qquad 1$$

Lemma 5.15. *Critical pairs resulting from overlapping between rule (3) and rule (3) resolve.*

$$\overline{t_l}\,\overline{t_j}\,\overline{t_{l'}}$$

$$(3) \quad \swarrow \qquad\qquad \searrow \quad (3)$$

Proof. $\overline{s_k}\,\overline{t_{l'}} \quad \hookrightarrow \qquad\qquad \overline{t_l}\,\overline{s_{k'}} \qquad\qquad \square$

$$(4)$$

Lemma 5.16. *Critical pairs resulting from overlapping between rule (5) and rule (3) resolve.*

$$\overline{s_m}\,\overline{t_j}\,\overline{t_{l'}}$$

$$(5)\quad \swarrow \qquad\qquad \searrow \quad (3)$$

$$\overline{t_p}s_{m'}\overline{s_k}\,\overline{t_{l'}} \qquad\qquad\qquad \overline{s_m}\,\overline{s_{k'}}$$

Proof. $\qquad\downarrow\quad(4)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

$$\overline{t_p}s_{m'}\overline{t_l}\,\overline{s_{k'}} \qquad\qquad\qquad\qquad 1$$

$$\downarrow\quad(\beta)$$

$$\overline{t_p}t_p\overline{s_m}\,\overline{s_{k'}} \qquad\qquad \hookrightarrow$$

Lemma 5.17. *Critical pairs resulting from overlapping between rule* (β) *and rule* (3) *resolve.*

$$s_{m'}\overline{t_l}\,\overline{t_j}$$

$$(\beta)\quad \swarrow \qquad\qquad \searrow \quad (3)$$

$$t_p\overline{s_m}\,\overline{t_j} \qquad\qquad\qquad s_{m'}\,\overline{s_k}$$

Proof. $\qquad\downarrow\quad(5)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

$$t_p\overline{t_p}s_{m'}\overline{s_k} \qquad\qquad \hookrightarrow$$

Using the same argument as in claim 3.4, we obtain that if $\Phi_{m'}$ is in $\mathfrak{R}'$

$$(1_{l',q,q'}) : t_{l'} \to \overline{t_q}s_{q'}$$

then the set of rules $\Phi_{m''}$ is also in $\mathfrak{R}'$, where $\Phi_{m''}$ is: $\quad (2_{l',q,q'}) : s_{q'}\overline{t_{l'}} \to t_q$

$$(3_{l',q,q'}) : \overline{t_{l'}}\,\overline{t_q} \to \overline{s_{q'}}$$

and the following rule of kind (4) $: \overline{s_{k'}}\,\overline{t_q} \to \overline{t_j}\,\overline{s_{q'}}$ is added to $\mathfrak{R}'$.

Lemma 5.18. *Critical pairs resulting from overlapping between rule* (4) *and rule* (3) *resolve.*

$$\overline{s_k}\,\overline{t_{l'}}\,\overline{t_q}$$

$$(4) \quad \swarrow \qquad \searrow \quad (3)$$

Proof. $\quad \overline{t_l}\,\overline{s_{k'}}\,\overline{t_q} \qquad\qquad\qquad \overline{s_k}\,\overline{s_{q'}} \qquad\qquad \square$

$$\downarrow \quad (4)$$

$$\overline{t_l}\,\overline{t_j}\,\overline{s_{q'}} \qquad\qquad \hookrightarrow$$

$$(3)$$

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